Robotics Research Technical Report

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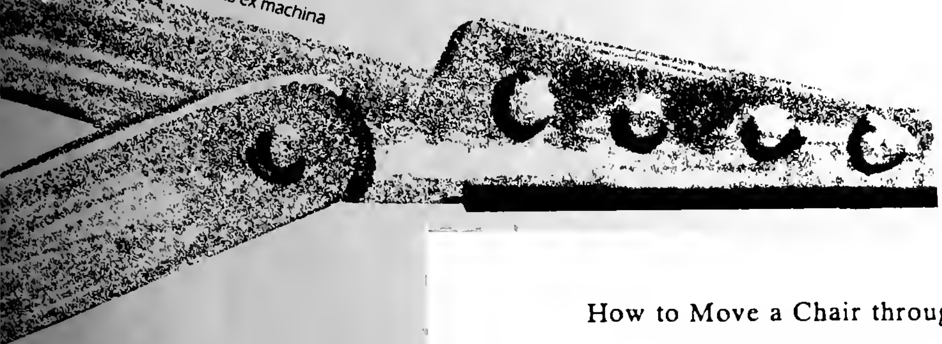

How to Move a Chair through a Door

by

Chee K. Yap

Technical Report No. 238
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ABSTRACT

We define the *door-width* of a simple polygon (a 'chair') and give an $O(n^2)$ algorithm for computing its door-width. It is first shown that all passages of the chair through the door can be reduced to a sequence of certain elementary motions. We introduce the technique of constraint analysis in characterizing elementary motions. Our algorithm actually constructs a motion of the chair through a door, and thus is a 'local-expert' for planning motion through doors. Such algorithms have applications in more general motion-planning systems in robotics.

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1. Introduction

To move a given chair through a door, a natural question is how wide the door should be. The minimum width of a passable door is called the width of the chair. The question was first raised by Gilbert Strang [Str82] who studied the case of convex chairs in two and three dimensions.[1] We focus on the polygonal chairs in this paper but allow non-convex chairs. Although the notion of width for convex chairs is easily captured, the width of non-convex chairs admits several reasonable interpretations. We will note three possibilities in this paper.

To compute the width of a chair, we in fact construct a motion of the chair through any door of sufficient width. Thus our problem can be regarded as an instance of the motion-planning problem that has been extensively studied in the robotics literature. We refer to [Yap85] for a survey. It should be observed that our problem is a curious one from the robotics point of view: in robotics, one is typically interested in algorithms for moving some *fixed* robot B through an *arbitrary* environment E (from some well-defined class of environments) where E is described as part of the problem instance. In our case, we have an algorithm for a *fixed* environment E but permit the robot B to be *arbitrarily* specified (from the class of polygonal robots). In [Yap85] we term the former *basic planning problems* and the latter *local planning problems*. A motion-planning algorithm that is specialized for a fixed locality E is thus a 'local expert'. Although the papers [HJW82],[KK84] were not originally so conceived, we can regard them as further examples of local planning.

We indicate the pragmatic significance of distinguishing between basic versus local planning. It is known (see references in [Yap85]) that, even for the simplest classes of robots, when the degrees of freedom of the robots is arbitrarily large then motion-planning is probably intractable (NP-hard or worst). Even if the number of degrees of freedom is fixed (e.g. rigid polygonal robots), the complexity of motion-planning of arbitrarily complex robots seems to require high degree polynomials; this complexity is

[1] The interested reader may refer to the Strang's paper for the amusing genesis of the problem

unacceptable in practice. However, a practical motion-planning system can exploit the heuristic fact that 'most' robot motion takes place in relatively uncluttered space. In uncluttered space, the robot B can be approximated by some simpler enclosing shape B'. For example if we enclose B in the minimal size disk B', then fast $O(n \log n)$ algorithms are available. At the isolated localities that are cluttered, and only then, we invoke more sophisticated algorithms that take the detail shape of B into account. A further key observation is that most of these cluttered localities are reasonably stereo-typed, such as a door or a corner in a corridor. This is especially true in a controlled or planned environment such as a factory floor. We should be able to provide local experts for such stereo-type situations. This paper is therefore one contribution towards providing such local experts.

The rest of this paper is organized as follows: section 2 gives some preliminary concepts. In sections 3 and 4 we define three notions of width. We call these the door-width, ring-width and elastic ring-width, respectively. This is partly to indicate that for non-convex polygons, the concept of width is not unique and can be subtle. The main technical results of the paper are in sections 5, 6 and 7 where we develop an algorithm for computing the door-width of a polygon. We conclude in section 8. There is an appendix containing some geometric lemmas.

2. Preliminaries

In this paper, we want to refer to a 'physical' chair (a polygonal body) moving in the plane: by 'physical' we mean that the chair is a geometric figure that has no fixed or instantiated position in the plane. To make this idea precise, we must distinguish between two coordinate systems (or 'frames of reference'): the *standard frame* and the *actual frame*. Points in standard (resp. actual) frame will be designated as *standard* (resp. *actual*) *points*.

A *chair* (or *body*) B is a compact set of standard points whose boundary $\partial(B)$ is a simple closed polygonal path. Throughout this paper, we fix some arbitrary B and let n be the number of corners of B. Typically but not necessarily, the origin of the standard frame is a point in B. Note that in general, if we treat several bodies then each body must be given its own standard frame.

It is important to note that there are only conceptual but no formal distinctions between the standard and actual points. However, we do not mix concepts from the two frames. For instance if x is a standard point and Y is a set of actual points, the expression '$x \in Y$' is not meaningful. Or again, if X is a set of standard points, we would not use the expression '$X \cap Y$'. Instead, the connection between the two frames is always done via certain transformations, as we now explain: A *placement* (of the standard frame in the actual frame) is represented by a rigid transformation of the standard frame, specified by a triple $Z = (x, y, \theta)$ where (x, y) is a pair of reals and $0 \leq \theta < 2\pi$. The placement Z specifies that the origin of the standard frame is at the actual point (x, y)

and the positive x-axis of the standard system is to be given the orientation θ.[2] More precisely, for any standard point p, the transformation $Z = (x, y, \theta)$ of p is the actual point defined by

$$p[Z] = (x + p_x\cos\theta - p_y\sin\theta,\ y + p_x\sin\theta + p_y\cos\theta)$$

which, in matrix notation, is

$$= (x, y) + p \cdot cis(\theta)$$

where $cis(\theta)$ is a 2×2 rotation matrix. If S is a set of standard points then $S[Z] = \{p[Z] : p \in S\}$. For instance $B[Z]$ denotes the set of actual points occupied by B at placement Z. Also let $\theta(Z)$ denote the orientation of Z.

The $[Z]$ and $<Z>$ *notations*: For each Z, we can regard $[Z]$ as a map from the standard frame to the actual frame of reference It is convenient to have a notation for the inverse map: if p is an actual point, define $p<Z>$ to be the standard point q where $q[Z] = p$. Thus $<Z>$ is the inverse of the map $[Z]$.

The set $\Re$ of all placements is essentially $E^2 \times S^1$ where E^2 is the Euclidean plane and S^1 the unit circle. ($\Re$ is also called the Euclidean group or rigid planar transformations.) We can make $\Re$ into a metric space $\Re(B)$ by defining the distance $d(Z, Z')$ between two placements Z, Z' to be the Hausdorff distance between $B[Z]$ and $B[Z']$. Note the dependence of this metric on B.

Let W be a set of actual points called the *obstacle set*. We define Z to be *free (for B, with respect to W)* if

$$interior(B[Z]) \cap W = \emptyset.$$

(Note: in the literature, the alternative condition $B[Z] \cap W = \emptyset$ is sometimes used to define free placements.) The *free space* of B with respect to W is the set of Z that are free; let $FP(W, B) \subseteq \Re(B)$ denote this free space. If B is understood, we write $FP(W)$ instead of $FP(W, B)$. A *motion* of B amidst obstacle W is a continuous path $\pi : [0, 1] \to \Re(B)$. The motion is *free* or (*avoids* the obstacle W) if $\pi(t) \in FP(W, B)$ for all t. We sometimes denote $\pi(t)$ by Z_t and write $\pi = \{Z_t : t \in [0, 1]\} \subseteq \Re(B)$. A *constant motion* π is one in which $\pi(t) = \pi(0)$ for all t.

Given two motions π_0 and π_1 where $\pi_0(1) = \pi_1(0)$, we define their *composition* π by:

$$\pi(t) = \begin{cases} \pi_0(2t) & \text{if } t \leq 1/2 \\ \pi_1(2t-1) & \text{if } t \geq 1/2 \end{cases}$$

[2] A vivid picture comes from thinking of the actual frame as the table on which we place a sheet of paper (the standard frame) that has a drawing of the chair

We denote this composition by $\pi = \pi_0;\pi_1$.

Two motions π_0 and π_1 are said to be *similar* if there is a continuous monotonic function $f : [0, 1] \to [0, 1]$ such that $f(0) = 0, f(1) = 1$ and $\pi_0(t) = \pi_1(f(t))$ for all t. Otherwise, π_0 and π_1 are *dissimilar*. For instance, it is not hard to see that $\pi_0;(\pi_1;\pi_2)$ and $(\pi_0;\pi_1);\pi_2$ are similar provided the compositions in these expressions are well-defined. We distinguish motions only up to similarity, and hence we will be happy not to parenthesize a sequence of compositions: the above will be written as $\pi_0;\pi_1;\pi_2$.

3. The Door Width

We formalize the first notion of the width of a body B. Let $w > 0$ and consider the following set of actual points:

$$D_w = \{(0, y) : y \leq 0 \text{ or } y \geq w\} \subseteq E^2$$

We call D_w a *door of width* w. This captures the notion of an infinite wall with a hole of width w. It is convenient to have a notation for the following partition of the actual plane by the y-axis:

$$LHS = \{(x, y) : x < 0\}, \quad (\text{'left hand side'})$$

$$RHS = \{(x, y) : x > 0\}, \quad (\text{'right hand side'})$$

$$Y = \{(x, y) : x = 0\}, \quad (\text{'y-axis'})$$

We say a motion $\{Z_t : t \in [0, 1]\}$ of B is a *passage through* D_w if it avoids D_w and also

$$B[Z_0] \subseteq LHS \quad \text{and} \quad B[Z_1] \subseteq RHS.$$

In other words, the body moves from the left half-plane to the right half while avoiding collision with the wall.

Definition. The *door-width* of B is the smallest w such that there is a passage through D_w.

This is equivalent to the definition of Strang in case B is convex. The door-width of a convex body is simple to compute: let (v, e) denote a (vertex, edge) pair of B. We say (v, e) is a *opposite pair* if the line through p and parallel to e is a supporting line of B.

Lemma 1.

(a) *The door-width of B is attained at some opposite pair (v, e) in the sense that the door-width is equal to the distance between v and the line through e*

(b) *There are $O(n)$ opposite pairs and these can be obtained in linear time*

Part (b) is easy, and part (a) follows from Strang's characterization [Str82] that the width of any convex shape S is obtained as the projection of S onto some line. The natural extension of this characterization to 3 dimensions is false, and this is one of the

points of Strang's paper. The algorithmic implication of this lemma is that we need not rotate a convex chair inside the doorway.

It follows that the door-width of convex bodies can be computed in linear time. Now consider the following body B_0 consisting of a $2 \times (2 - \epsilon)$ rectangle with a $1 \times \epsilon$ slit on one of the sides of length $2 + \epsilon$. Here $1 \gg \epsilon > 0$ and it is not hard to see that B_0 has a door-width of 2:

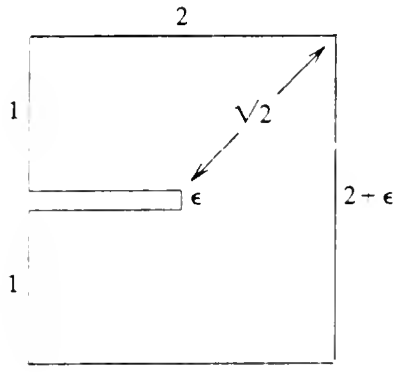


Figure 1. Body B_0

4. The Ring Width and Elastic Width

The purpose of this section is to show that the notion of width for non-convex objects admits other reasonable interpretations. This section may be omitted since later sections are independent of it. The reader may intuitively feel that the width of B_0 in the previous example could be defined so that it is $\sqrt{2}$ rather than 2. This intuition comes from imagining a 'ring' of diameter $\sqrt{2}$ that is passed over the body B_0. However, the correct definition of what it means for a body to pass through a ring is a little subtle (the reader is invited to attempt such a definition). For instance, a suitable definition must contend with the following example: a ring of width 1 can superficially sweep every point of the following non-convex quadrilateral even though the ring cannot pass through the body:

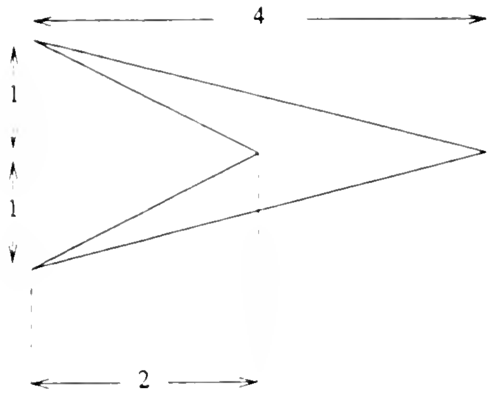


Figure 2. Non-convex quadrilateral

To capture the concept precisely, first define the *ring* of diameter $w > 0$ to be the pair of actual points

$$R_w = \{(0, 0), (0, w)\}.$$

It is also convenient to have a notation for the (actual) line segment

$$I_w = \{(0, y) : 0 \le y \le w\} \subseteq E^2.$$

Consider a motion $\pi = \{Z_t : t \in [0,1]\}$ of B that avoids R_w such that $B[Z_0] \subseteq LHS$ and $B[Z_1] \subseteq RHS$. For any standard point $p \in interior(B)$, let $J_p \subseteq [0, 1]$ be the set of time instances t such that $p[Z_t]$ lies in I_w. Note that well-known properties of the unit interval imply that J_p is a finite union of disjoint intervals of the form

$$J_p = [s_1, t_1] \cup [s_2, t_2] \cup, \ldots, \cup [s_m, t_m], \quad (m \ge 1)$$

where

$$0 < s_1 \le t_1 < s_2 \le t_2 < \cdots < s_m \le t_m < 1.$$

If t is one of the t_i's, and s is one of the s_i's, define

$$\chi^-(s) = \chi^-(s, \pi) = \begin{cases} 1/2 & \text{if } p[Z_{s-}] \subseteq LHS \\ -1/2 & \textit{otherwise} \end{cases}$$

$$\chi^-(t) = \chi^-(t, \pi) = \begin{cases} 1/2 & \text{if } p[Z_{t-}] \subseteq RHS \\ -1/2 & \textit{otherwise} \end{cases}$$

The notation Z_{s-} in the preceding definition refers to the placement of B at some moment preceding, but 'sufficiently close' to, t. Similarly for Z_{t-}. Now we define the π-

crossing number of $p \in interior(B)$ (relative to π) to be

$$\chi(p) = \chi(p, \pi) = \sum_{i=1}^{m} \chi^{-}(s_i) + \sum_{i=1}^{m} \chi^{-}(t_i).$$

Essentially, the crossing number of p increments (resp. decrements) by one each time p crosses from *LHS* (resp. *RHS*) to the *RHS* (resp. *LHS*). It is easy to see that since B moves from *LHS* to *RHS* that the crossing number of any interior point p must be an integer value.

Definition. A motion $\pi = \{Z_t : t \in [0, 1]\}$ is a *passage through* R_w if it avoids R_w and

(1) $B[Z_0] \subseteq LHS$ and $B[Z_1] \subseteq RHS$, as before.

(2) For each point p in $interior(B)$, the π-crossing number of p is ≥ 1.

It can be shown that for any passage π, the π-crossing number does not depend on the choice of the interior point p (use the fact that B is connected).

Definition. The *ring-width* of B is the smallest w such that there is a passage through R_w.

It is now easy to see that the ring-width of B_0 in the last example is indeed $\sqrt{2}$.

There is one question which, though obvious for door-width, is not clear for ring-width. Does the ring-widths satisfy the following *monotonicity property*: if there is a passage of B through a ring of diameter w then there is a passage of B through any ring of diameter w' where $w' > w$.

Conjecture 1. Ring-widths have the monotonicity property.

Another possible notion of width is what we call an *elastic width*. This is essentially the same as the ring-width except that we have an elastic ring that (can continuously) shrink to a smaller width if necessary. We omit a formal definition. It is clear that for any body B,

$$door\text{-}width(B) \geq ring\text{-}width(B) \geq elastic\text{-}width(B).$$

It is clear that these two inequalities are in fact equalities for a convex B. Our example B_0 above shows that the first inequality may be strict. We conjecture that elasticity does not help:

Conjecture 2. For any chair B, $ring\text{-}width(B) = elastic\text{-}width(B)$.

5. Constraint analysis

Recall that Y is the y-axis of the actual plane. For any $Z \in \Re(B)$, if $interior(B[Z]) \cap Y \neq \emptyset$ we say that Z has *positive clearance*, otherwise it has *zero clearance*. If Z has positive clearance, there is a unique pair of standard points (u, v) such

$$interior(B[Z]) \cap Y \subseteq (u[Z], v[Z])$$

and the distance between u and v is minimum. Assuming that $u[Z]$ lies above $v[Z]$ on Y, we use the notation

$$top<Z> \qquad \text{and} \qquad bot<Z>$$

to refer to these points u and v, respectively. If Z has zero clearance, $top<Z>$ and $bot<Z>$ are undefined.

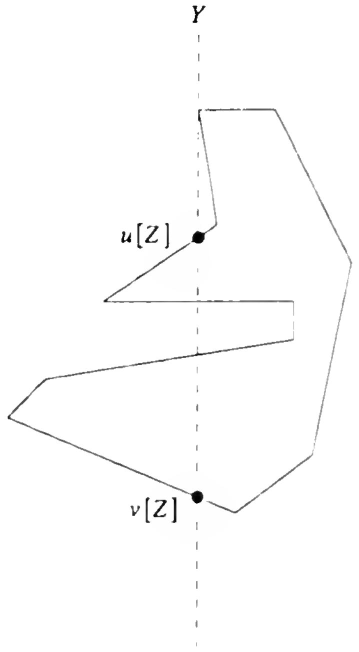


Figure 3. Illustrating the definition of clearance

For any placement Z, define $Clearance(Z)$ to be 0 if Z has zero clearance; otherwise $Clearance(Z)$ is the distance between $top<Z>$ and $bot<Z>$. For any motion π, let $Clearance(\pi)$ be the maximum of $Clearance(\pi(t))$, $t \in [0, 1]$.

Two placements Z and Z' are *equivalent* if they are vertically translatable to each other: more precisely, if $Z = (x, y, \theta)$ and $Z' = (x', y', \theta')$ then $x = x'$ and $\theta = \theta'$. Two motions π and π' are *equivalent* if some motion π'' similar to π is pointwise equivalent to π'', i.e., for all t, $\pi''(t)$ and $\pi'(t)$ are equivalent. Despite the asymmetry in this definition, the relation is a true mathematical equivalence relation. Note that two equivalent placements or motions have the same clearance. Again, we normally distinguish motions up to equivalence.

Definition. A pair (e, e') of distinct edges of B is called a *bridge* if there exists a placement Z with positive clearance such that

(a) $Y<Z>$ contains no vertices of B and

(b) $top\langle Z\rangle$ and $bot\langle Z\rangle$ (resp.) lie in e and e'.

We say such a placement Z is *on the bridge* (e, e'). Let $S \subseteq \Re(B)$ be the set of all placements on the bridge (e, e'). The closure of S in $\Re(B)$ is called the *channel* of (e, e') and denoted $C(e, e')$.

Observe that because of (b) in the definition of a bridge, we regard (e, e') and (e', e) as distinct bridges. Indeed, bridges and hence channels always come in such pairs. Let $V(e, e')$ denote the set of vertices of B that lies outside the interior of the rectangle (or a triangle in the degenerate case) formed by the endpoints of e and e'. Thus the 3 or 4 distinct endpoints of e and e' are included in $V(e, e')$. If Z is on the bridge (e, e') then the y-axis $Y\langle Z\rangle$ partitions $V(e, e')$ into two non-empty sets L and R. It is not hard to see that (L, R) and (e, e') each uniquely the other. Hence we sometimes denote the channel $C(e, e')$ by $C(R, L)$. Further observe that a channel forms a connected subset of $\Re(B)$.

Form the respective convex hulls H_L and H_R of L and R. A point p on the (boundary of the) convex hull H_L or H_R is said to be *visible* in the channel $C(L, R)$ if there is a placement Z in the channel such that $p[Z]$ lies on the y-axis, i.e., $p \in Y\langle Z\rangle$. We conventionally arrange for edges e, e' of a bridge to have an east-west orientation with H_L on the west and H_R on the east. The set of visible points of H_L (resp. H_R) form a contiguous chain on (the boundary of) H_L (resp. H_R) which we call the *west bank* (resp. *east bank*). A placements $Z \in C(e, e')$ is said to be *on the east bank* if $Y\langle Z\rangle$ intersects the east bank. Similarly for the west bank. It is easy to see that the boundary of a channel (as a subset of $\Re(B)$) is composed of precisely the placements on the two banks.

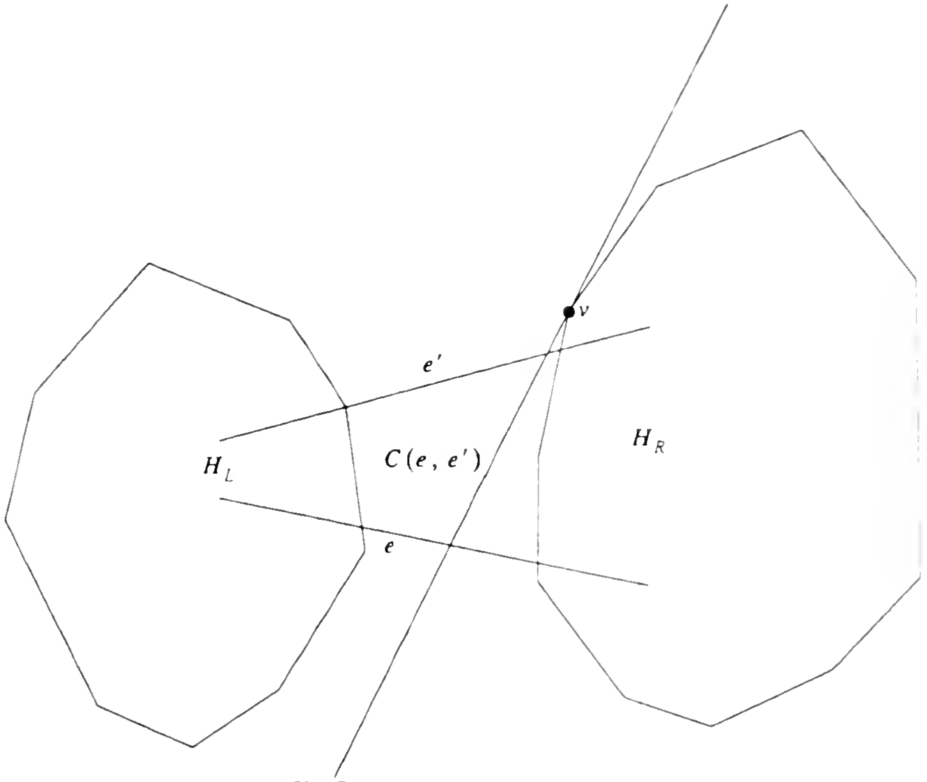


Figure 4. Vertex v lies on the east bank of the (e, e') channel

Our main problem in this section is show how to obtain canonical motions in channels.

Definition. Let $\pi = \{Z_t : t \in [0,1]\}$ be a motion of B. We say π is a *pure motion* if it has one of these two forms:

(i) π is a *pure rotation* if there exists a standard point p such that for all t, $p[Z_t]$ is constant and $\theta(Z_t)$ is monotonic (increasing or decreasing) in t. The point p is called the *center* of the rotation.

(ii) π is a *pure translation* if $\theta(Z_t)$ is a constant for all t and for each standard point p, $p[Z_t]$ moves monotonically in a straight line.[3]

We would like to further reduce pure motions in a channel to a very simple form. Translations have simple geometry so the main work lies in studying pure rotations in a channel. To do this, consider the following setup: fix a channel $C(e, e')$ and let π be a pure rotation in the channel centered at a visible vertex v of the channel. We are

[3] It is enough that any two standard points $p \neq p'$ move in straight lines, since transformations are rigid

interested in the clearance function $D(t) = Clearance(\pi(t))$. In the appendix, we show

Lemma 2. *The function $D(t)$ has at most one maxima t $t_v \in (0, 1)$*

If t_v exists, we say that the orientation of the line $Y<\pi(t_v)>$ is *special* for v relative to (e, e').

To define the pure motions of interest we introduce the auxiliary concept of 'constraints'.

Definition. Let $C(e, e')$ be a channel. A *constraint* K is a syntactic (i.e. formal) object associated with a channel $C(e, e')$ with one of the following four forms:

$$K(e),\ K(e'),\ K(v),\ K(u, v),\ K(v, e, e')$$

where u,v are visible vertices of the channel $C(e, e')$. We say:

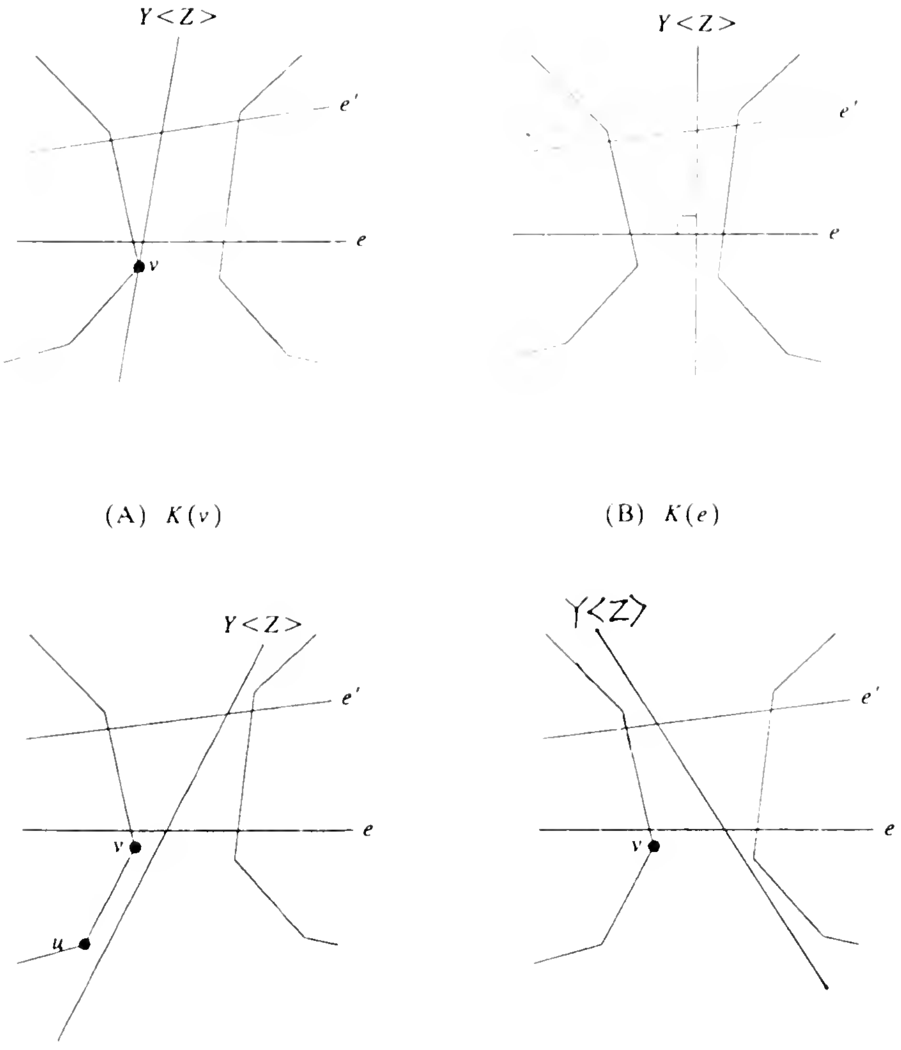
(A) Z satisfies $K(v)$ if $v[Z] \in Y$ and v is a visible vertex in $C(e, e')$.

(B) Z satisfies $K(e)$ if the y-axis Y is perpendicular to $e[Z]$. Similarly for $K(e')$.

(C) Z satisfies $K(u, v)$ if Y is parallel to the line $u[Z]v[Z]$ where u, v are adjacent vertices on one bank of the channel.

(D) Z satisfies $K(v, e, e')$ if the orientation of $Y<Z>$ is the special orientation of v relative to the bridge (e, e').

Each constraint K is associated with a predicate (also denoted K) on the channel where the predicate K holds at Z if Z satisfies K according to the preceding definition. We often confuse the distinction between the constraint and its predicate.



(A) $K(v)$ (B) $K(e)$

(C) $K(u, v)$ (D) $K(v, e, e')$

Figure 5. Four types of constraints

A constraint K is *trivial* if the corresponding predicate is identically false. (Clearly no constraint can be identically true.) For instance, suppose the number of visible vertices of a channel is k. Note that $k \geq 3$. Then there are k nontrivial constraints of type (A), at most 2 nontrivial constraints of type (B), $k - 2$ nontrivial constraints of type (C), and at most k nontrivial constraints of type (D).

Two constraints K_1 and K_2 of a channel are *dependent* if they are identical as predicates, otherwise they are *independent*. The next lemma indicates that constraints of

satisfy some other constraint K of the form $K(u', v')$ or $K(e)$ or $K(u, e, e')$. We will charge the critical placement Z to K. Note that $K(u', v')$ can be charged $O(1)$ times, and there are $O(n^2)$ constraints of the form $K(u', v')$. The lemma holds for this case. Next, each $K(e)$ is charged $O(n)$ times. To see this, each e is involved with at most n other edges to form channels. But each channel causes $K(e)$ to be charged $O(1)$ times. Since there are overall n such choices for $K(e)$, the result also follows. Finally, each $K(u, e, e')$ is charged $O(1)$ times; it remains to show that there are $O(n^2)$ such constraints. Each constraint of the form $K(u, e, e')$ can be charged to u; it is easy to see that u can be charged only $O(n)$ times and there are $O(n)$ choices for u. The proof is now complete. **Q.E.D.**

Definition. A pure motion $\pi = \{Z_t : t \in [0, 1]\}$ is *semi-elementary* if the following holds:

(i) If π is a translation then there exists a constraint K of the form $K(e)$ or $K(u, v)$ or $K(u, e, e')$ such that for all t, Z_t satisfies K. If π is a rotation then there exists a constraint of the form $K(v)$ such that for all t, Z_t satisfies K.

(ii) Z_t is not critical for all $0 < t < 1$.

If, in addition, Z_0 and Z_1 are critical then we say that π is *elementary*.

One special type of elementary translation π deserves special mention: if $\pi(t)$ is equivalent to $\pi(0)$ for all t, we call π a *slide*.

It is not hard to see that for any two elementary motions if their initial and final placements are respectively equivalent then they are equivalent motions. In other words, up to equivalence, elementary motions may be identified with the equivalence classes of their initial and final placements. This shows an upper bound of $O(n^4)$ elementary motions. We improve this estimate considerably.

Theorem 6. *There are $O(n^2)$ inequivalent elementary motions.*

Proof. It is easy to see that each critical placement Z is the starting placement for at most 3 inequivalent elementary translations, namely, a slide and at most two translations (in the 2 directions normal to y-axis). The final placements of such elementary motions are completely determined. Thus the $O(n^2)$ bound holds in the case of elementary translations.

To bound the number of elementary rotations, we consider two cases of critical placements Z. If $Y\langle Z\rangle$ contains only one vertex of B (there must be at least one vertex) then it is easy to see that there are at most 2 elementary rotations starting at Z. Each such rotation can be charged to Z. Each such Z is charged at most twice and there are $O(n^2)$ choices for such Z. The second case is where $Y\langle Z\rangle$ contains more than one vertex of B. Suppose $Y\langle Z\rangle$ contains the vertices $u_1, \ldots, u_k$, for some $k \geq 2$, occurring in the order indicated. Each elementary rotation centered at u_i ($i = 1, \ldots, k$) and starting from such a Z can be charged to a consecutive pair of vertices, either (u_i, u_{i-1}) or

(u_{i-1}, u_i), arbitrarily. Clearly no pair (u, v) of vertices is charged more than 4 times and there are only $O(n^2)$ such pairs. Hence there are $O(n^2)$ inequivalent elementary rotations overall. **Q.E.D.**

6. How to cross a channel

A *nice* motion π is either a semi-elementary motion or else that can be written in the form

$$\pi = \pi_0;\pi_1; \cdots ;\pi_k \quad (k \geq 2)$$

where

(i) π_0 and π_k are semi-elementary rotations,

(ii) $\pi_1, \ldots, \pi_{k-1}$ are elementary motions (translations or rotations).

(iii) No other motions from $\pi(0)$ to $\pi(1)$ achieves a smaller clearance.

For example, elementary motions are nice. Note that $\pi_0(1)$ and $\pi_k(0)$ are necessarily critical.

We show that any motion can be reduced to a nice one. We begin by considering motion in a fixed channel $C(e, e')$. As usual, we talk about the east and west banks of the channel. We define a map

$$Im : C(e, e') \rightarrow C(e, e')$$

that takes any placement $Z \in C(e, e')$ by a direct translation to a placement $Im(Z)$ on the west bank. More precisely, for any Z, let v be the vertex on the west bank that is closest to $Y<Z>$. If d is the distance of v from $Y<Z>$ and $Z = (x, y, \theta)$ then $Im(Z) = (x-d, y, \theta)$. In particular, $v \in Y<Im(Z)>$.

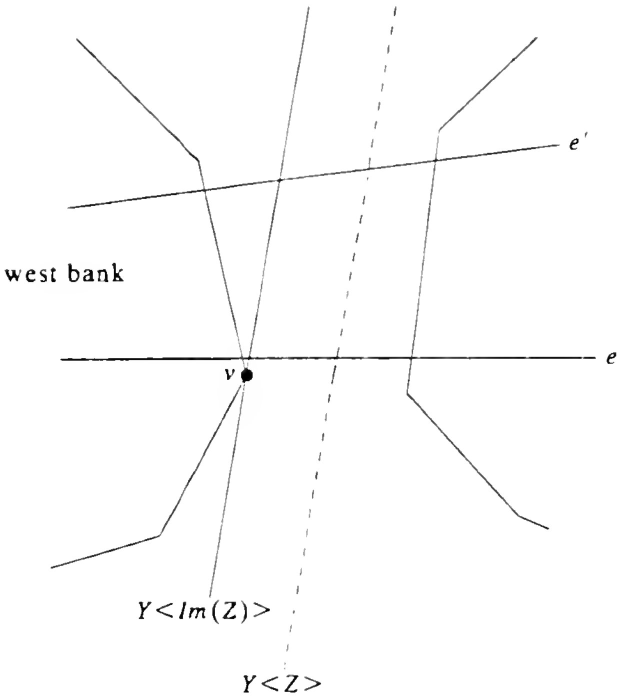


Figure 6. The map *Im*

We may assume that the bridge is narrower on the west bank than the east bank, i.e., $Clearance(Z) \geq Clearance(Im(Z))$. Given any motion π in a channel, we define a motion $Im(\pi)$ on the west bank in the natural way.

Lemma 7. *Let Z_0 be on the east bank of a channel. Then there is a nice motion π from Z_0 to $Im(Z_0)$ that achieves its maximum clearance at Z_0.*

Proof. We construct $\pi = \pi_0;\pi_1;\pi_2$ as follows:

π_0: Rotate Z_0 in the direction of decreasing clearance until the first critical placement Z_1. (If Z_0 is critical, then π_0 is a constant motion and $Z_0 = Z_1$.)

π_1: Translate from Z_1 across the channel to $Im(Z_1)$. This motion is elementary. Note that $\pi_0;\pi_1$ has non-increasing clearance.

π_2: Finally, rotate from $Im(Z_1)$ directly to $Im(Z_0)$. This motion is semi-elementary and hence achieves its maximum clearance at $Im(Z_1)$ or at $Im(Z_0)$.

Since the clearance at $Im(Z_1)$ and $Im(Z_0)$ are both no larger than the clearance at Z_0, we conclude that π attains its maximum clearance at Z_0. **Q.E.D.**

Lemma 8. *Let Z_0 and Z_1 be on the banks of a channel. Then there is a nice motion π from Z_0 to Z_1.*

Proof. There is an obvious nice motion π_1 from $Im(Z_0)$ to $Im(Z_1)$ consisting of rotations only. Furthermore, no motion from Z_0 to Z_1 has clearance less than

$$\max\{Clearance(Z_0),\ Clearance(Z_1),\ Clearance(\pi_1)\}$$

The previous lemma gives us nice motions π_0 and π_2 from Z_0 to $Im(Z_0)$ and from Z_1 to $Im(Z_1)$ (respectively). Since the clearance of π_0 and π_2 are attained at Z_0 and Z_1 (respectively) it follows that $\pi_0;\pi_1;\pi_2$ achieves the minimum clearance of any motion from Z_0 to Z_1. The two consecutive semi-elementary rotations in $\pi_0;\pi_1$ can be replaced by an elementary rotation so that the result is a nice motion π_3. Repeating this for $\pi_3;\pi_2$, we get the desired result. Q.E.D.

We are now ready for our main result:

Theorem 9. *Let $Z_0 \in LHS$, $Z_1 \in RHS$. If there is a passage of B from Z_0 to Z_1 through a door D_w then there is one of the form.*

$$\pi_0;\pi_1;\pi_2$$

where π_0 and π_2 have zero clearance and are each the composition of at most three pure motions, π_1 is a composition of elementary motions that achieves the minimum clearance (i.e. the door-width of B).

Proof. Suppose π is a passage of B from Z_0 to Z_1 whose clearance is minimum. We may suppose that there is an open interval $I \subseteq [0, 1]$ such that $\pi(t)$ has zero clearance precisely when $t \notin I$. We may further suppose that for some $k \geq 1$, there are moments

$$s_0 \leq t_0 < s_1 \leq t_1 < \cdots < s_k \leq t_k$$

such that $I = (s_0, t_k)$ and for all $t \in [0, 1]$, $\pi(t)$ satisfy a constraint of type (A) iff $t \in [s_i, t_i]$ for some $i = 0, \ldots, k$.

These s_i's and t_i's divide π into portions which we now analyze in turn. First, we know by the previous lemma that each portion of the form $\pi[t_i, s_{i+1}]$ $(i = 0, \ldots, k-1)$ can be replaced by a nice motion. Next consider $\pi[s, t]$ for some $(s, t) = (s_i, t_i)$, $i = 0, \ldots, k$. It is not hard to see that $\pi[s, t]$ can also be replaced nice motion. Using the same trick as before of replacing two consecutive semi-elementary rotations by an elementary rotation, we get a nice motion π_3 from $\pi_3(0) = \pi(s_0)$ to $\pi_3(1) = \pi(t_k)$.

Since π_3 is nice, we may write is as $\pi_4;\pi_5;\pi_6$ where π_5 is a composition of elementary motions, π_4 and π_6 are semi-elementary rotations, and $\pi_4(1)$ and $\pi_6(0)$ are critical. It is now easy to give a motion π_0 (resp. π_2) composed of at most three pure

motions (translation, rotation and translation) from Z_0 to $\pi_4(1)$. (resp. $\pi_6(0)$ to Z_1). With π_1 set to π_5, we see that $\pi_0;\pi_5;\pi_2$ satisfies the theorem. **Q.E.D.**

7. Algorithmic details

The basic idea is to construct a weighted graph whose edges correspond to elementary motions. We then run a shortest path algorithm (Djikstra's) on the graph to discover a minimum clearance path.

For any simple polygon B, let $G(B)$ be the node-weighted (undirected) graph where each node represent an equivalence class of critical placements and whose edges (or *arcs*)[4] correspond to elementary motions that connect adjacent critical placements. (The slides would be represented by self-loops but we may ignore explicit consideration of them here.) The weight attached to a node is equal to the clearance of the critical placement represented by the node. Call a node *initial* (resp. *final*) if its critical placements Z satisfy $B[Z] \subseteq LHS$ (resp. $B[Z] \subseteq RHS$). In particular, initial and final nodes have zero weights. We further introduce into $G(B)$ two special nodes called the *source* and the *sink* and introduce arcs connecting the source to all the initial placements and connecting the sink to all the final placements. The weights of the source and sink are zero.

The number of nodes and number of arcs in $G(B)$ is $O(n^2)$ where B has n vertices. Given any path specified as a sequence $(u_1, \ldots, u_k)$ of nodes in $G(B)$, let the weight of the path be the maximum of the weights of u_i ($i = 1, \ldots, k$). From the main theorem in the previous section, the width of the chair B is the minimum weight path connecting the source and the sink. The minimum weight path can be found by Djikstra's algorithm in $O(n^2)$ steps.

It remains to show how to construct $G(B)$ in $O(n^2)$ steps. For each vertex v of B we rotate a ray R about v, keeping track of the sequence of edges that intersect the rotating line. In actual fact, it is sufficient to keep track of the first and last edges intersected by R (these edges may be undefined). As will be shown below, this list of first and last edges for each v is sufficient for us to construct $G(B)$. The full circle of 2π radians is thereby broken up into *basic intervals* during which the rotating ray R does not pass through any other vertices. This process is easily accomplished in $O(n\log n)$ per vertex or $O(n^2\log n)$ overall. However, using the results in [Wel85],[AAGHI86], this bound can be improved to $O(n^2)$ time. For completeness, we briefly show how to do this *assuming* that we have computed for each vertex v of B, the (circular) list $L(v)$ of all the other vertices in the order encountered by the ray R sweeping a full circle about v. It is known [CL85],[AAGHI86] that the n lists $L(v)$ can be obtained in $O(n^2)$ time. For simplicity in this discussion, we assume that no three vertices are collinear.

---

[4] To avoid confusion between the 'edges' of the polygon B and the 'edges' of $G(B)$, we call the latter *arcs*. Similarly, 'vertices' refer B and 'nodes' refer to $G(B)$.

We first break up $L(v)$ into two sublists $L_A(v)$ and $L_B(v)$ (resp.) consisting of vertices encountered by the ray R above and below (resp.) the horizontal line through v. Let $\Theta_A(v)$ and $\Theta_B(v)$ be the corresponding list of orientations of the ray R through vertices in $L_A(v)$ and $L_B(v)$, respectively. For any orientation θ, we let $pred\,(\theta)$ (resp. $succ_v(\theta)$) to refer to the largest (resp. smallest) orientation $\theta' \in \Theta(v)$ where $\theta' < \theta$ (resp. $\theta < \theta'$). For example, each 'basic interval' defined above is of the form $(\theta, succ_v(\theta))$ where $\theta \in \Theta(v)$. For each $\theta \in \Theta(v)$ we define $f_v(\theta)$ to be the first edge in B seen by the ray from v at any orientation strictly between θ and $succ_v(\theta)$. If this edge is undefined then $f_v(\theta) = \infty$. Let $\Phi_A(v)$ be the list of $\{f_v(\theta) : \theta \in \Theta_A(v)\}$ corresponding to $\Theta_A(v)$.

Lemma 10. *The lists $\Phi_A(v)$ for all vertices v can be computed in $O(n^2)$ time.*

Proof. Suppose the ray from v to another vertex u is at orientation $\theta \in \Theta_A(v)$. Then it is not hard to see $f\,(\theta)$ is equal to one of the following:

(a) $f_v(pred_v(\theta))$

(b) One of the edges incident to u.

(c) $f_u(pred_u(\theta))$.

Furthermore, it is easy to check in constant time which is the case. Let $v_1, v_2, \ldots, v_n$ be the vertices of B in order of decreasing y-coordinate. We will compute the lists $\Phi_A(v_i)$ for $i = 1, 2, \ldots, n$ (in the indicated order). Using an $O(n\log n)$ precomputation (using a plane-sweep method), it is seen that for each list $\Phi_A(v_i)$ we can determine the first value in the list. To obtain successive entries in the list, the above cases (a)-(c) apply. Cases (a) and (b) are easy; in case (c), it is clear that we must lookup the list $\Phi_A(v_j)$ for some $j = 1, \ldots, i-1$.

We claim that with $O(n^2)$ preprocessing, the value $f_{v_j}(pred_{v_j}(\theta))$ in case (c) can be obtained in constant time. To see this, we see that $\theta \in \Phi_A(v_i)$ corresponds to $\theta + \pi \in \Phi_B(v_j)$. Using an $O(n^2)$ preprocessing, we may assume that there is a pointer link from the occurence of θ in $\Phi_A(v_i)$ to the occurence of $\theta - \pi$ in $\Phi_B(v_j)$. Next, for each $\theta' \in \Phi_B(v_j)$, we may assume there is a pointer link to $pred_{v_j}(\theta' - \pi)$ in $\Phi_A(v_j)$; such links can be obtained in linear time by a merge-like process on $\Phi_A(v_j)$ and $\Phi_B(v_j)$. The above two types of links shows that case (c) can also be handled in $O(1)$ time. This concludes our demonstration of the lemma. **Q.E.D.**

We now show how to detect all the bridges (e, e'). Let u and v be adjacent vertices on one of the banks of $C(e, e')$ with u and v lying on opposite sides of the bridge (see figure).

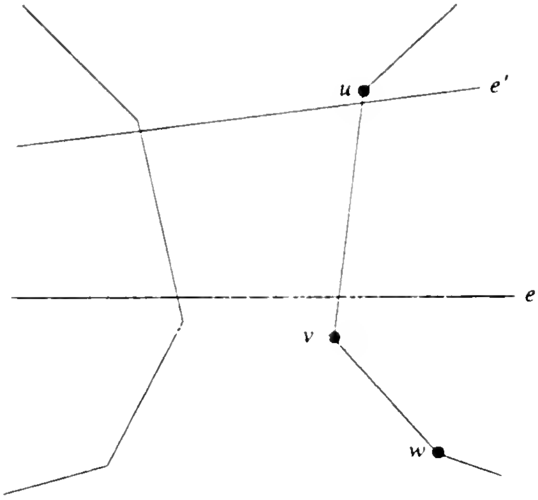


Figure 7. Detecting the bridge (e, e')

Suppose the orientation from v to u is θ. Then (up to interchanging the roles of u and v, etc) the four following conditions hold:

(i) $f_v(\theta) = e$. Furthermore, let u' be the point such that the orientation from v to u' is $succ_v(\theta)$. Then all of the edges incident to v lies on the side of the line uv opposite to u'.

(ii) $f_u(\theta) = \infty$.

(iii) $f_u(pred_u(\theta - \pi)) = e'$. The edges incident to u must lie on the same side of the line uv as the edges incident to v, as described in (i).

(iv) $f_v(pred\ (\theta - \pi)) = \infty$

This test can be done for each pair (u, v) for a total time of $O(n^2)$.

For each bridge (e, e'), we claim that the above process may be modified to discover its left and right banks. Referring to figure 7, consider two adjacent vertices v, w on one bank of $C(e, e')$ with v, w on the same side of the bridge. It is easy to set up conditions on $\Phi(v)$ and $\Phi(w)$ similar to (i)-(iv) which detect that v and w are adjacent vertices on one bank of a channel. Hence, by linking together these adjacent pairs vertices on each banks of a channel, we get a representation of the banks as adjacency lists. We leave the details to the reader.

Finally, we determine all the arcs of $G(B)$ -- this is equivalent to identifying all elementary motions. We first refine the basic intervals as follows: by merging the orientations of $\Theta_A(v)$ with $\{\pi - \theta : \theta \in \Theta_B(v)\}$, each basic interval in $\Theta_A(v)$ is broken up into *subbasic* intervals. Similarly we can break up the basic intervals of $\Theta_B(v)$ into subbasic intervals.

Each subbasic interval I of v corresponds to a rotation in some channel $C(e, e')$. If v is a visible vertex of the channel, then clearly this rotation is a composition of elementary rotations in $C(e, e')$. We show how to further break up such I into elementary rotations. It is easy to subdivide I into smaller intervals at those orientations where constraints of the form $K(e)$, $K(e')$ or $K(v, e, e')$ are satisfied. It is slightly more involved to break up I at orientations where it satisfies $K(u, u')$ where u, u' are adjacent vertices on the bank of (e, e') opposite v. To do this, we process each channel separately. Take the two banks (represented as linked lists) of a channel and by a process similar to merging two sorted lists, we can obtain the orientations in each subbasic interval I corresponding to constraints of the form $K(u, u')$. This completes the subdivision of the basic intervals into elementary rotations. Furthermore, the elementary translations across that channel can be obtained during the same merge process. This concludes our proof of the main algorithmic result:

Theorem 11. *The graph $G(B)$ can be constructed $O(n^2)$ steps and the minimum weight path can be found in $O(n^2)$ time.*

Corollary. *The door-width of a simple n-gon can be computed in $O(n^2)$ time. The same algorithm can also constructs a motion of the polygon through any door of sufficient width.*

Remark. The above algorithm may be sped up in practice as follows. A point p on the boundary of a polygon P is said to be *visible* if there is a ray originating at p that does not intersect the interior of P. The set of visible points of P is composed of a set of polygonal path-segments where each path-segment is a connected subset of the boundary of P. By connecting these path-segments in the natural way, we obtain what may be called the *visible hull* of P. Clearly, $door\text{-}width(P) = door\text{-}width(VH(P))$ where $VH(P)$ denotes the visible hull of P. It follows easily from [LL84] that $VH(P)$ can be computed in $O(n \log n)$ time. Hence, it may be more efficient to replace the polygon by its visible hull before running the door algorithm.

8. Conclusion

(1) Our introduction of constraint analysis here appears to be a general and powerful tool in studying geometric problems of this nature. It is intimately related to retraction approach of motion planning [Yap85]. Planned future papers will illustrate other uses of such analysis.

(2) Can our $O(n^2)$ complexity for door-width be improved to subquadratic bounds? Our algorithm is optimal among those which must construct the graph $G(B)$. This is because, in the worst case, $G(B)$ has size $\omega(n^2)$. For instance, the following polygon has $omege(n^2)$ channels and elementary motions: take a regular convex n-gon and at each vertex 'add an arrowhead' (see figure)

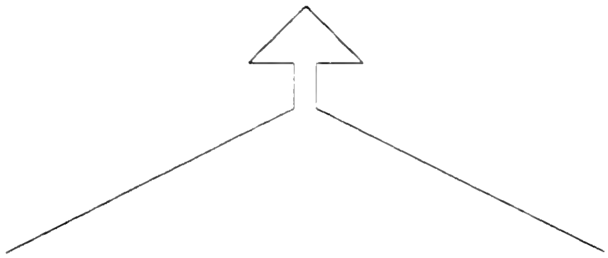


Figure 8. Arrowhead

In any case, subquadratic bounds may be possible for special chairs (eg. rectilinear ones).

(3) It would be nice to prove the two conjectures concerning ring-widths in section 4. We expect some of the ideas of the present paper can be extended to give an algorithm for computing the ring-width.

(4) Several issues are raised by the practical approach to motion-planning suggested in our introduction: First, how does one partition an environment E into parts which are 'clutter-free' and parts that are 'cluttered', assuming some suitable definitions for these intuitive terms. We refer to [SS85] for one notion of clutter-free environments. Next, given a method of partitioning, we need a systematic cataloguing (stereotyping) of the tight situations. Finally, for those common stereotypes, we need efficient local algorithms. Besides the door stereotype of this paper, another interesting situation is a corner in a corridor or hallway. In this context, we refer to several interesting articles on the so-called 'moving sofa' problem cited in [Str82].

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APPENDIX: Behavior of the clearance function during a rotation

Our goal is to prove lemma 2 in section 4. Let $C(L, R) = C(e, e')$ be a channel. We investigate the clearance during a pure rotation π in the channel. Let the vertex $v \in L \cup R$ be the center of the rotation. We may suppose that e and e' are not parallel, since otherwise the analysis would be trivial. Let Λ, Λ' be the (standard) rays originating from a common point O and containing e, e' respectively. Thus v lies outside the wedge $\Lambda\Lambda'$.

Choose O to be the origin, Λ to be the positive x-axis, Λ' to lie above the x-axis. Wlog, v lies below the x-axis and to the right of Λ'. Let Q (resp. Q') be the foot of the perpendicular from v to the line containing Λ (resp. Λ'). For any angle θ, let R_θ denote the ray emanating from v at an angle θ measured counter-clockwise from the ray $v \to Q'$. In particular, R_0 denotes the ray $v \to Q'$ and $R_{-\alpha}$ denotes the ray $v \to Q$. See the figure.

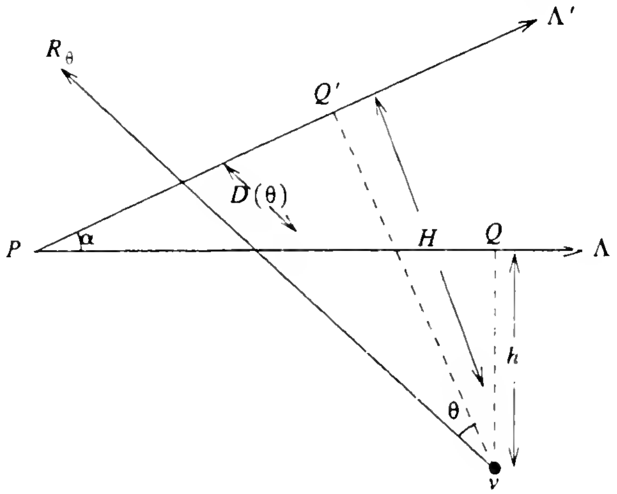

Figure 8. Case $\alpha \leq \pi/2$

Let Θ denote the range of orientations where R_θ, $\theta \in \Theta$, intersects both Λ and Λ'. Clearly the pure rotation π may be parametrized by $\theta \in \Theta$. Define $D(\theta)$ to be the distance between the two points of intersection of R_θ with Λ and Λ'. Clearly lemma 2 is a consequence of the following.

Lemma. *The function $D(\theta)$ is either monotonic or else it has a unique maxima and a unique minima in its domain Θ of definition.*

Proof. Let $\alpha > 0$ be the angle between Λ and Λ'. First assume that

$$0 < \alpha < \pi/2$$

as in the above figure. Let $H = |Q' - v|$ be the distance from v to line containing Λ' and $h = |Q - v|$ be the distance of v below the x-axis. Then it is easy to see that

$$D(\theta) = \frac{H}{\cos\theta} - \frac{h}{\cos(\theta - \alpha)}. \tag{1}$$

Clearly $D(\theta) \geq 0$ for $\theta \in \Theta$. Let $\Theta = [\theta_1, \theta_2]$. By breaking the analysis into three parts if necessary, we may assume that one of the following three cases hold:

(i) $-\pi/2 < \theta_1 < \theta_2 < -\alpha$

(ii) $-\alpha \le \theta_1 < \theta_2 \le 0$

(iii) $0 < \theta_1 < \theta_2 < \pi/2 - \alpha$.

We shall use primes (e.g. D') to denote differentiation with respect to θ. We have

$$D'(\theta) = \left(\frac{H}{\cos\theta}\right)\tan\theta - \left(\frac{h}{\cos(\theta+\alpha)}\right)\tan(\theta+\alpha). \tag{2}$$

In cases (i) and (ii), we show that $D'(\theta) < 0$ so that D is strictly monotonic: (i) If $-\pi/2 < \theta < -\alpha$, this follows from the fact that $\frac{H}{\cos\theta} \ge \frac{h}{\cos(\theta+\alpha)} \ge 0$ (since $D(\theta) \ge 0$) and $\tan\theta < \tan(\theta+\alpha) < 0$. (ii) If $-\alpha \le \theta \le 0$, then we see that $\tan\theta \le 0 \le \tan(\theta+\alpha)$, so again D' is negative.

Case (iii) is unfortunately more intricate. Setting

$$F(\theta) = \frac{\sec(\theta+\alpha)\tan(\theta+\alpha)}{\sec\theta\tan\theta} \tag{3}$$

we may write

$$D'(\theta) = \sec\theta\tan\theta[H - hF(\theta)]. \tag{4}$$

We will show that

$$F''(\theta) > 0. \tag{5}$$

This implies that $F(\theta)$ has at most two zeroes. The expression (4) then shows that $D'(\theta)$ also has at most two zeroes. We show that $D'(\theta)$ cannot have exactly one zero. This follows from the fact that $D'(0) < 0$ (from (2)) and $D'(\theta_2) < 0$ (since, by definition, $D(\theta_2) = 0$).

We can now see the global behaviour of $D(\theta)$: $D(\theta)$ is monotonically decreasing in the range (i) and (ii), and possibly after an interval of increase in range (iii), it finally decreases to zero. Hence the lemma.

To show (5), we take logarithms in (3),

$$\log F = [\log\sec(\theta+\alpha) + \log\tan(\theta+\alpha)] - [\log\sec\theta + \log\tan\theta].$$

Differentiating:

$$\frac{F'}{F} = [\tan(\theta+\alpha) - \tan\theta] + [\sec(\theta+\alpha)\operatorname{cosec}(\theta+\alpha) - \sec\theta\operatorname{cosec}\theta]. \tag{6}$$

Differentiating again,

$$(F'/F)' = \frac{F''F - (F')^2}{F^2}$$

$$= [\sec^2(\theta+\alpha) - \sec^2\theta] + [(\sec^2(\theta+\alpha) - \operatorname{cosec}^2(\theta+\alpha)) - (\sec^2\theta - \operatorname{cosec}^2\theta)]$$

$$\frac{F''}{F} = \left(\frac{F'}{F}\right)^2 + 2[\sec^2(\theta+\alpha) - \sec^2\theta] + [\operatorname{cosec}^2\theta - \operatorname{cosec}^2(\theta+\alpha)] \tag{7}$$

Since $F(\theta) > 0$ (from (3)) and it is easy to see that the RHS of (7) is positive, this concludes our proof of (5).

It remains to consider the case where

$$\alpha > \pi/2.$$

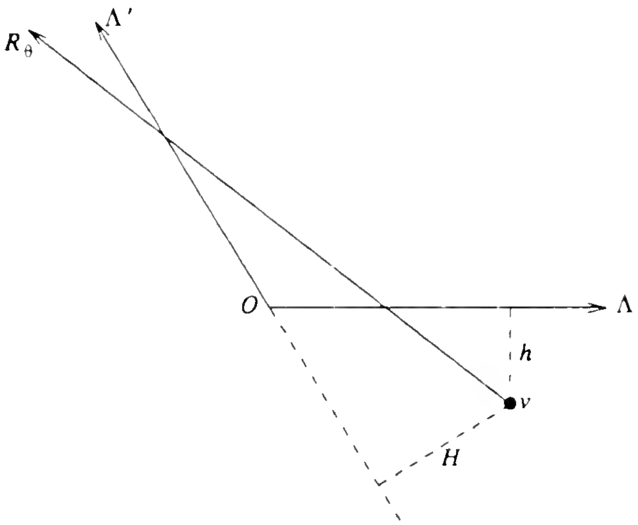

Figure 9. case $\alpha > \pi/2$

The expression (1) for $D(\theta)$ still holds except that the range Θ now satisfies

$$-\pi/2 < \theta < 0 < (\theta + \alpha) < \pi/2$$

for $\theta \in \Theta$. Hence $\tan\theta < 0 < \tan(\theta + \alpha)$ and we conclude $D'(\theta) < 0$ as in case (ii) above. **Q.E.D.**

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